Matsusaka's Theorem

X will be non-singular complex projective variety of dimension n.

Matsusaka's Theorem

Let L be an ample divisor on X.
Then there exists M depending only on the Hilbert polynomials

$\chi(X, \mathcal{O}_X(mL))$ such that for $m \geq M$.

mL is very ample.

. $\boxed{\text{Remark}}$ A refinement by Kollár and Matsusaka. states that one can get M to depend only on (L^n) and $(K.L^{n-1})$

This is the version we will prove.

Main Theorem

Let L be an ample divisor, and B nef divisor on X.

Set $\rho_L := (L^n)$, $\rho_B := (B.L^{n-1})$, $\rho_K := (K.L^{n-1})$

Then there exists

$$M_0 = M_0(\rho_K, \rho_L, \rho_B).$$

such that $mL - B$ is nef for any $m \geq M_0$.

Now, the idea will be first to see how to get Matsusaka's theorem from this Theorem. For this we will need some lemmas and previous theorems.

Arithmetic Lemmas

- **Criterion for very ample bundles**

Let B be an ample and free line bundle and N a line bundle such that

$$H^i(X, N \otimes B^{\otimes -i}) = 0 \quad \text{for } i > 0.$$

Then $N \otimes B$ is very ample.

- **Theorem of Angehrn and Siu**

Let L be an ample divisor on X. $x \in X$ a fixed point. and assume that

$$(L^{\dim Z} \cdot Z) > \binom{n+1}{2}^{\dim Z}$$

For every irreducible variety $Z \subseteq X$ passing through x.
Then $K_X + L$ is free at x. i.e. $\mathcal{O}_X(K_X + L)$ has a section which does not vanish at x.

- **Lemma 1** There exist integers λ_n, β_n depending only on n such that:

$$\lambda_n K_X + \beta_n L + P$$

is very ample. for any L ample and P nef.

Proof of Lemma 1 | As L is ample.

$$\left(\left(\binom{n+1}{2}+1\right)L\right)^{\dim Z}\cdot Z > \left(\binom{n+1}{2}+1\right)^{\dim Z} \quad \text{for any } Z \leq X.$$

Hence $K_X + \left(\binom{n+1}{2}+1\right)L$ is free by Angehrn–Siu

as free + ample is ample:

$B := K_X + \left(\binom{n+1}{2}+2\right)L$ is ample (and free).

Now, as $(n+1)B + P + (-j)B$ is ample for

$0 < i \leq n$.

by Kodaira vanishing we get

$$H^i\Big(X, \mathcal{O}_X\big(\underbrace{K+(n+1)B+P}_{\text{"}N\text{"}} + (-i)\underbrace{B}_{\text{"}B\text{"}}\big)\Big) = 0 \quad \text{for } i>0.$$

We have the conditions for our criterion for very ampleness.

Hence $K_X + (n+1)B + P + B$

$$= \underbrace{(n+3)K_X + (n+2)\left(\binom{n+1}{2}+2\right)L}_{\text{this we will call } A.} + P$$

is very ample.

<u>Corollary</u> Given L ample, B nef. There exists an integer $M_1 = M_1(\rho_L, \rho_K, \rho_B)$ such that $mL - B$ is very ample for any $m \geq M_1$.

<u>Proof</u> Applying the main theorem to L, $B+A$.

We get $M_1 = M_0(\rho_L, \rho_K, \rho_{(A+B)})$ such that $mL - B - A$ is nef. By the property of A

$$\underset{\text{nef}}{A + P} = A + \underset{\text{nef}}{(mL - B - A)} \quad \text{is very ample.}$$

and $\rho_{(A+B)}$ depends only on ρ_B, ρ_L, ρ_K and n. as A is a linear comb. of L and K.

Now we will only be interested in showing the Main Theorem. For that a special construction will be quite useful.

Proof of the Main theorem.

We want to show that for every curve C, $(mL-B)^{n-1}\cdot C \geq 0$.

For any irreducible curve $C \subseteq X$. our objective will be to construct

$$\begin{tikzcd} C' \arrow[d, "\varphi"'] & \subseteq & Y' \arrow[d, "\mu"] & \\ C & \subseteq & Y & \subseteq X \end{tikzcd}$$

Here Y will vary depending on the C.

s.t.

- Y is an irreducible variety.
- $\mu : Y' \to Y$ is a resolution of singularities of Y.
- C' is an irreducible curve mapping finitely onto C.

And a number $\nu_p = \nu_p(\rho_L, \rho_K, \rho_B)$ depending only on $p := \dim Y$, ρ_L, ρ_K, ρ_B.

such that:

- There exists a positive integer $k = k_Y \leq \nu_p$ such that $\mathcal{O}_{Y'}(\mu^*(kL-B|_Y))$ has a section which does not vanish identically along C'. [Notice the dimension of Y depends on C.]

Assuming such a construction, we can prove the main theorem.

Proof of Main Theorem

$$\Rightarrow ((\nu_p L - B)\cdot C)) \geq ((kL - B)\cdot C))$$

$$= \frac{1}{\deg \varphi}(\mu^*(kL-B)\cdot C')$$

$$\geq 0.$$

Now putting $M = \max\{\nu_p \mid 1 \leq p \leq n\}$

Then $(mL-B)\cdot C \geq (\nu_p L - B)\cdot C \geq 0$. for any $m \geq M$. $\square$

Idea of the construction:

For the construction of Y (for a fixed C)

We want to proceed inductively to build a chain.

$$X = Y_n \supset Y_{n-1} \supset \dots \supset Y_p = Y \supseteq C.$$

of subvarieties of X.

Given Y_i we want to apply a non-vanishing criterion to get non-zero sections $s_i \in H^0(Y_i, \mathcal{O}_{Y_i}(kL - B))$ for suitable k. If s_i does not vanish on C, then we are done with $Y = Y_i$. Otherwise we take Y_{i-1} to be an irreducible component of $\mathrm{Zeroes}(s_i)$ containing C and continue.

Non-Vanishing criterion

- **Numerical criterion for bigness:**

Let D and E be ample divisors on X. They can be nef.
such that $D^n > n \cdot (D^{n-1} \cdot E)$.

Then $D-E$ is big.

Proof: as the hypothesis and the result are preserved by multiplying D and E by a large number, we can assume D and E to be very ample.
Fix $m>0$ and choose m general divisors $E_1, \dots, E_m \in |E|$ linearly equivalent to E. Then

$$\mathcal{O}_X(m(D-E)) \cong \mathcal{O}_X\left(mD - \sum_{i=1}^{m} E_i\right)$$

So $H^0(\mathcal{O}_X(mD-E))$ is identified with sections of $\mathcal{O}_X(mD)$ vanishing on each E_i.

Hence:

$$h^0(X, \mathcal{O}_X(m(D-E))) \geq h^0(X, mD) - \sum_{i=1}^{m} h^0(E_i, \mathcal{O}_{E_i}(mD))$$

$$= \frac{(D^n)}{n!} m^n - \sum \frac{(D^{n-1} \cdot E_i)}{(n-1)!} \cdot m^{n-1} + O(m^{n-1}) \quad \text{by asymp. R.R.}$$

$$= \underbrace{\frac{D^n}{n!} m^n - n \frac{(D^{n-1}) \cdot E}{n!} m^n}_{\text{positive multiple of } m^n} + O(m^{n-1}).$$

Hence $D-E$ is big.

Double-Point Formula:

Given $X \subset \mathbb{P}^N$ taking a general projection, we get a birational morphism $f : X \to X' \subseteq \mathbb{P}^{n+1}$, X' a hypersurface.

The locus where f is not an isomorphism is called the double-point divisor and is linearly equivalent to
$\boxed{(d-n-2)H - K_X}$, where H is a hyperplane section of X, and d the degree of X.

* Remember Nadel Vanishing.

Non-vanishing for rationally effective divisors.

Let H be divisor on X (dim. n) s.t.

$$H^0(X, \mathcal{O}_X(mH)) \neq 0 \quad \text{for some } m \quad \text{(i.e. non-negative Iitaka dimension)}$$

Let L be an ample divisor on X. Then there exists $1 \leq k \leq n+1$ s.t.

$$H^0(X, \mathcal{O}_X(K_X + H + kL)) \neq 0.$$

Proof) Choose $A \in |mH|$.

Due to Nadel vanishing.

$$H^i(X, \mathcal{O}_X(K_X + H + tL) \otimes \mathcal{J}(\tfrac{1}{m}A))$$

for $t \geq 1$.

Hence.

$$h^0(X, \mathcal{O}_X(K_X + H + tL) \otimes \mathcal{J}(\tfrac{1}{m}A)) = \chi(X, \mathcal{O}_X(K_X + H + tL) \otimes \mathcal{J}(\tfrac{1}{m}A))$$

for $1 \leq t \leq n+1$. As the right hand side is a polynomial on t of degree n. Hence for some $1 \leq t \leq n+1$.

$$H^0(X, \mathcal{O}_X(K_X + H + tL) \otimes \mathcal{J}(\tfrac{1}{m}A)) \neq 0.$$

The assertion follows as this is a subgroup of

$$H^0(X, \mathcal{O}_X(K_X + H + tL)). \quad \square$$

Lemma 2

Let $Y \subseteq X$ be an irreducible subvariety of X having dimension p, and let $\delta_Y := (A^p \cdot Y)$. For A as defined in Lemma 1

$\{A = \lambda_n K_X + \beta_n L$ such that $A + n e f$ is very ample$\}$

Set $\nu(Y) = p \cdot (L^{p-1} \cdot (B + \delta_Y A) \cdot Y) + (p+2)$.

and consider any resolution $\mu : Y' \to Y$. Then there exists a positive integer $k_Y \leq \nu(Y)$ s.t. $\mathcal{O}_{Y'}(\mu^*(k_Y L - B))$ has a non-zero section.

Proof

As kL, $B + \delta_Y A$ are ample. As long as

$$\boxed{k \cdot (L^p \cdot Y) \geq p \cdot (L^{p-1} \cdot (B + \delta_Y A) \cdot Y) + 1 =: \nu'}$$

$kL - B - \delta_Y A$ will be big (by our bigness criterion).

In particular if $k = \nu'$ we have that.

So, we get the condition for non-vanishing criterion.

$H = (kL - B - \delta_Y A)$, $L = L$.

$$\Rightarrow H^0(Y', \mathcal{O}_{Y'}(K_{Y'} + \mu^*(k'L + kL - B - \delta_Y A))) \neq 0$$

for $1 \leq k' \leq p+1$.

i.e. $H^0(Y', \mathcal{O}_{Y'}(K_{Y'} + \mu^*(k_Y L - B - \delta_Y A))) \neq 0$

$$k_Y \leq \nu' + p + 1 = \nu.$$

As we want $H^0(Y', \mathcal{O}_{Y'}(\mu^*(k_Y L - B))) \neq 0$

It would be enough to prove

$$H^0(Y', \mathcal{O}_{Y'}(-K_{Y'} + \mu^*(\delta_Y A)) \neq 0.$$

For this, consider the embedding $Y \hookrightarrow \mathbb{P}$ defined by $|A|$ and take a general projection to $\mathbb{P}^{p+1}$ mapping Y birationally onto a hypersurface of degree δ_Y. Composing with μ we have a morphism

$$f: Y' \to \mathbb{P}^{p+1}$$

birational onto its hypersurface image. By the doublepoint formula. The locus of double points of f is supported on an effective divisor lin. eq. to $(\delta_Y - p - 2)\mu^*(A) - K_Y$.

Hence $H^0(Y', \mathcal{O}_{Y'}((\delta_Y - p - 2)\mu^* A - K_{Y'}) \neq 0$

therefore $H^0(Y', \mathcal{O}_{Y'}(\delta_Y \mu^* A - K_{Y'}) \neq 0.$

as $\mu^* A$ is effective

$\square$

But here we get some $\nu(Y)$ depending on $L^{p-1} \cdot B \cdot Y$

$L^{p-1} \cdot A \cdot Y$

and $\underbrace{\delta_Y = A^p \cdot Y}$

This is particularly bad as it

Hodge Type inequality.

Given D, E nef. Then: for $1 \leq p \leq n$

$$\left(D^p \cdot E^{n-p}\right)\left(E^n\right)^{p-1} \leq \left(D \cdot E^{n-1}\right)^p$$

Corollary: Given D nef L ample

$$\left(D^p \cdot L^{n-p}\right) \leq \left(D \cdot L^{n-1}\right)^p$$

Construction of the chain of subvarieties

We want $X = Y_n \supset Y_{n-1} \supset \cdots \supset Y_p = Y \supseteq C$

with $\nu_i = \nu_i(\rho_L, \rho_K, \rho_B)$ s.t. $\mu_i^*(kL - B)$

has a section for $k = k_{Y_i} \leq \nu_i$

1st Apply Lemma 2 to X.

We get $k \leq \nu(X) = n \cdot (L^{n-1} \cdot (B + \delta_X A)) + (n+2)$

s.t. $H^0(X, \mathcal{O}_X(kL - B)) \neq 0$.

δ_X at this point does not depend on ρ_L, ρ_B, ρ_K.

Using the Hodge-type inequalities: $(p = n)$.

$$(A^n)(L^n)^{n-1} \leq (A \cdot L^{n-1})^n$$

Hence $A^n \leq (A \cdot L^{n-1})^n = \rho_A^n$. Hence we get

ν_n depending only on ρ_L, ρ_K, ρ_B.

and we get $H^0(X, \mathcal{O}_X(kL - B)) \neq 0$ for some $k \leq \nu_n$

Let $s_n \in H^0(X, \mathcal{O}_X(kL - B))$ be a non-zero section.

If s_n does not vanish on $C \subset X$, we take $Y = Y' = X$.

Otherwise

<u>2nd Step</u>) Take a component Y_{n-1} of Zeroes $(s_n) \subset X$ which contains C. Fix a resolution

$$\mu_{n-1} : Y'_{n-1} \to Y_{n-1}.$$

and an irreducible curve $C'_{n-1} \subseteq Y'_{n-1}$ lifting C.

We apply Lemma 2. to get.

$$K = K_{Y_{n-1}} \leq \nu(Y_{n-1})$$
$$= (n-1)\left(L^{n-2}(B + \delta_{Y_{n-1}} A) \cdot Y_{n-1}\right) + (n+1)$$

s.t. $\mu^*(kL-B)$ has a non-zero section.

Now, we need ν_{n-1} depending ρ_L, ρ_K, ρ_B bounding $\nu(Y_{n-1})$.

as A, L, B are nef:

- $\delta_{Y_{n-1}} = A^{n-1} \cdot Y_{n-1}$ as Y_{n-1} is a component of a divisor in $|kL-B|$.

$$\leq A^{n-1} \cdot (kL-B)$$
$$\leq A^{n-1} \cdot (\nu_n L - B)$$
$$\leq \nu_n (A^{n-1} \cdot L)$$
by Hodge-type inequality
$$\leq \nu_n (A \cdot L^{n-1})^{n-1}$$

- $L^{n-1} \cdot (Y_{n-1}) \leq (L^{n-1})(kL - B)$
$$\leq \nu_n (L^n)$$

- $(L^{n-2} \cdot B \cdot Y_{n-1}) \leq \nu_n \cdot L^{n-1} \cdot B$

- $(L^{n-2} \cdot A \cdot Y_{n-1}) \leq \nu_n (L^{n-1} \cdot A)$

as $A = \lambda_n K_X + \beta_n L$, everything depends only on ρ_L, ρ_K, ρ_B

And we get the required bound $\nu_{n-1}(\rho_L, \rho_K, \rho_B)$.

We again take $s_{n-1} \in H^0(Y'_{n-1}, \mathcal{O}_{Y'_{n-1}}(\mu_{n-1}^*(\kappa L - B)))$

with $\kappa \leq \nu_{n-1}$ the non-zero section.

- If it does not vanish identically on C'_{n-1}

  We take $Y' = Y_{n-1}$, $C' = C'_{n-1}$.

Otherwise we choose an irreducible component $\tilde{Y}_{n-2}$ of the Zeroes (s_{n-1}) containing C'_{n-1}.

Set $Y_{n-2} = \mu_{n-1}(\tilde{Y}_{n-2})$ and fix a resolution

$\mu_{n-2} : Y'_{n-2} \to Y_{n-2}$. and an irred. curve

$C'_{n-2} \subseteq Y'_{n-2}$ lifting C.

We again bound the intersection numbers with Y_{n-2} by $\rho_L, \rho_\kappa, \rho_B$ using Hodge-type inequalities.

This keeps going until the s_p does not vanish identically on C'_p.

Final Remark

Demaily estimates more carefully to get:

$$M_1(\rho_L, \rho_K, \rho_B) = \frac{(2n)^{\left(\frac{3^{n-1}-1}{2}\right)} \cdot \left(L^{n-1}\cdot(B+H)\right)^{\left(\frac{3^{n-1}+1}{2}\right)} \cdot \left(L^{n-1}\cdot H\right)^{\frac{3^{n-2}(2n-3)-1}{4}}}{\left(L^n\right)^{\frac{3^{n-2}(2n-1)-1}{4}}}$$

where $H = (n^3 - n^2 - n - 1)(K_X + (n+2)L)$.